On the numbers $e^e$, $e^{e^2}$ and $e^{e^{e^2}}$

by

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Abstract. We give two measures of simultaneous approximation by algebraic numbers, the first one for the triple $(e, e^e, e^{e^2})$ and the second one for $(\pi, e, e^{e^2})$. We deduce from these measures two transcendence results which had been proved in the early 70’s by W.D. Brownawell and the author.

Introduction

In 1949, A.O. Gel’fond introduced a new method for algebraic independence, which enabled him to prove that the two numbers $2^{\sqrt{2}}$ and $2^{\sqrt{3}}$ are algebraically independent. At the same time, he proved that one at least of the three numbers $e^e$, $e^{e^2}$, $e^{e^{e^2}}$ is transcendental (see [G] Chap. III, ).

At the end of his book [S] on transcendental numbers, Th. Schneider suggested that one at least of the two numbers $e^e$, $e^{e^2}$ is transcendental; this was the last of a list of eight problems, and the first to be solved, in 1973, by W.D. Brownawell [B] and M.Waldschmidt [W 1], independently and simultaneously. For this result they shared the Distinguished Award of the Hardy-Ramanujan Society in 1986. Another consequence of their main result is that one at least of the two following statements holds true:

(i) The numbers $e$ and $\pi$ are algebraically independent
(ii) The number $e^{e^{e^2}}$ is transcendental

Our goal is to shed a new light on these results. It is hoped that our approach will yield further progress towards a solution of the following open problems:

(?) Two at least of the three numbers $e$, $e^e$, $e^{e^2}$ are algebraically independent.

(?) Two at least of the three numbers $\pi$, $e$, $e^{e^2}$ are algebraically independent.

Further conjectures are as follows:

(?) Each of the numbers $e^e$, $e^{e^2}$, $e^{e^{e^2}}$ is transcendental

(?) The numbers $e$ and $\pi$ are algebraically independent.

We conclude this note by showing how stronger statements are consequences of Schanuel's conjecture.

1. Heights

Let $\gamma$ be a complex algebraic number. The minimal polynomial of $\gamma$ over $\mathbb{Z}$ is the unique polynomial

$$f(X) = \alpha_0 X^d + \alpha_1 X^{d-1} + \ldots + \alpha_d \mathbb{Z}[X]$$

which vanishes at the point $\gamma$, is irreducible in the factorial ring $\mathbb{Z}[X]$ and has leading coefficient $\alpha_0 > 0$. The integer $d = \deg f$ is the degree of $\gamma$, denoted by $[\mathbb{Q}(\gamma) : \mathbb{Q}]$. The usual height $H(\gamma)$ of $\gamma$ is defined by

$$H(\gamma) = \max \{ |\alpha_0|, |\alpha_1|, \ldots, |\alpha_d| \}$$
$$H(\gamma) = \max\{|a_0|, \ldots, |a_d|\}.$$ 

It will be convenient to use also the so-called **Mahler's measure** of $\gamma$, which can be defined in three equivalent ways. The first one is

$$M(\gamma) = \exp\left(\int_0^1 \log |f(e^{2\pi it})| dt\right).$$

For the second one, let $\gamma_1, \ldots, \gamma_d$ denote the complex roots of $f$, so that

$$f(X) = a_0 \prod_{i=1}^d (X - \gamma_i).$$

Then, according to Jensen's formula, we have

$$M(\gamma) = |a_0| \prod_{i=1}^d \max\{1, |\gamma_i|\}.$$ 

For the third one, let $K$ be a number field (that is a subfield of $\mathbb{C}$ which is a $\mathbb{Q}/$-vector space of finite dimension $[K:\mathbb{Q}]/$) containing $\gamma_i$ and let $M_K$ be the set of (normalized) absolute values of $K$. Then

$$M(\gamma) = \prod_{v \in M_K} \max\{1, |\gamma|_v\}^{[K_v: \mathbb{Q}]}$$

where $K_v$ is the completion of $K$ for the absolute value $v$ and $Q_v$ the topological closure of $Q$ in $K_v$ and $[K_v : \mathbb{Q}_v]$ the local degree.

Mahler's measure is related to the usual height by

$$2^{-d}H(\gamma) \leq M(\gamma) \leq \sqrt{d + 1}.H(\gamma).$$

From this point of view it does not make too much difference to use $H$ or $M$, but one should be careful that $d$ denotes the exact degree of $\gamma$, not an upper bound. We shall deal below with algebraic numbers of degree $d$ bounded by some parameter $D$.

**Definition.** For an algebraic number $\gamma$ of degree $d$ and Mahler's measure $M(\gamma)$, we define the **absolute logarithmic height** $h(\gamma)$ by

$$h(\gamma) = \frac{1}{d} \log M(\gamma).$$

### 2. Simultaneous Approximation

We state two results dealing with simultaneous Diophantine approximation. Both of them are consequences of the main result in [W 2]. Details of the proof will appear in the forthcoming book [W 3].

**2.1. Simultaneous Approximation to $e$, $e^e$ and $e^{e^2}$**
Theorem 1. There exists a positive absolute constant $c_1$ such that, if $\gamma_0, \gamma_1, \gamma_2$ are algebraic numbers in a field of degree $D$, then
\[
|e - \gamma_0| + |e^e - \gamma_1| + |e^{e^2} - \gamma_2| > \exp\{-c_1 D^2 (h_0 + h_1 + h_2)^{1/2} (h_1 + h_2)^{1/2} (h_0 + \log D) (\log D)^{-1}\}
\]
where $h_i = \max\{e, h(\gamma_i)\} \ (i = 0, 1, 2)$.

2.2. Simultaneous Approximation to $\pi, e$ and $e^{e^2}$

Theorem 2. There exists a positive absolute constant $c_2$ such that, if $\gamma_0, \gamma_1, \gamma_2$ are algebraic numbers in a field of degree $D$, then
\[
|\pi - \gamma_0| + |e^e - \gamma_1| + e^{e^{e^2}} - \gamma_2| > \exp\{-c_2 D^2 (h_0 + \log(Dh_1 h_2))^{1/2} h_1^{1/2} h_2^{1/2} (\log D)^{-1}\}
\]
where $h_i = \max\{e, h(\gamma_i)\} \ (i = 0, 1, 2)$.

3. Transcendence Criterion

3.1 Algebraic Approximations to a Given Transcendental Number

The following result is Théoréme 3.2 of [R-W 1]; see also Theorem 1.1 of [R-W 2].

Theorem 3. Let $\theta \in \mathbb{C}$ be a complex number. The two following conditions are equivalent:
(i) the number $\theta$ is transcendental.
(ii) For any real number $h \geq 10^2$, there are infinitely many integers $d \geq 1$ for which there exists an algebraic number $\gamma$ of degree $d$ and absolute logarithmic height $h(\gamma) \leq h$ which satisfies
\[
0 < |\theta - \gamma| \leq \exp(-10^{-7} h d^2).
\]

Notice that the proof of $(\bar{z}^2) \Rightarrow (\bar{z})$ is an easy consequence of Liouville's inequality.

3.2. Application to $e^e$ and $e^{e^2}$

Corollary to Theorem 1. One at least of the two numbers $e^e, e^{e^2}$ is transcendental.

Proof of the corollary. Assume that the two numbers $e^e, e^{e^2}$ are algebraic, say $\gamma_1$ and $\gamma_2$. Then, according to Theorem 1, there exists a constant $c_3 > 1$ such that, for any algebraic number $\gamma$ of degree $\leq D$ and height $h(\gamma) \leq h$ with $h \geq e$,
\[
|e - \gamma| > \exp\{-c_3 D^2 h^{1/2} (h + \log D) (\log D)^{-1}\}.
\]
We now use Theorem 3 for $\theta = e$ with $h = 10^{16} c_3^2$ and derive a contradiction.

4. Algebraic Independence
4.1 Simultaneous Approximation

The proof of the following result is given in [R-W 1], Théoréme 3.1, as a consequence of Theorem 3 (see also [R-W 2] Corollary 1.2).

**Corollary to Theorem 3.** Let \( \theta_1, \ldots, \theta_m \) be complex numbers such that the field \( \mathbb{Q}(\theta_1, \ldots, \theta_m) \) has transcendence degree 1 over \( \mathbb{Q} \). There exists a constant \( c > 0 \) such that, for any real number \( h \geq c \), there are infinitely many integers \( D \) for which there exists a tuple \( (\gamma_1, \ldots, \gamma_m) \) of algebraic numbers satisfying

\[
[\mathbb{Q}(\gamma_1, \ldots, \gamma_m):\mathbb{Q}] \leq D, \quad \max_{1 \leq i \leq m} h(\gamma_i) \leq h
\]

and

\[
\max_{1 \leq i \leq m} \{|\theta_i - \gamma_i|\} \leq \exp(-c^{-1}hD^2).
\]

4.2. Application to \( \pi, e \) and \( e^{\pi^2} \)

**Corollary to Theorem 2.** One at least of the two following statements is true:
(i) The numbers \( e \) and \( \pi \) are algebraically independent.
(ii) The number \( e^{\pi^2} \) is transcendental.

**Remark.** This corollary can be stated in an equivalent way as follows:

For any non constant polynomial \( P \in \mathbb{Z}[X] \), the complex number

\[ e^{\pi^2} + iP(e, \pi) \]

is transcendental.

The idea behind this remark originates in [R].

**Proof of the Corollary.** Assume that the number \( e^{\pi^2} \) is algebraic. Theorem 2 with \( \gamma_2 = e^{\pi^2} \) shows that there exists a constant \( c_4 > 0 \) such that, for any pair \( (\gamma_0, \gamma_1) \) of algebraic numbers, if we set

\[
D = [\mathbb{Q}(\gamma_0, \gamma_1):\mathbb{Q}] \quad \text{and} \quad h = \max\{e, h(\gamma_0), h(\gamma_1)\},
\]

then

\[
|\pi - \gamma_0| + |e - \gamma_1| > \exp\{-c_4D^2(h + \log D)^{1/2}h^{1/2}(\log D)^{-1}\}
\]

Therefore we deduce from the Corollary to Theorem 3 that the field \( \mathbb{Q}(\pi, e) \) has transcendence degree 2.

5. Schanuel's Conjecture

The following conjecture is stated in [L] Chap. III p. 30: (The results of this section are based on the conjecture to be stated).

**Schanuel's Conjecture.** Let \( \mathbb{F}_1, \ldots, \mathbb{F}_n \) be \( \mathbb{Q} \)-linearly independent complex numbers. Then, among the \( 2^n \)
The 7 numbers
\[ e, \pi, e^e, e^{e^2}, e^{e^{e^2}}, 2^{\sqrt{2}}, 2^{\sqrt{3}} \]
are algebraically independent.

Let us deduce from Schanuel's Conjecture the following statement (which is an open problem):

(?) The 7 numbers
\[ e, \pi, e^e, e^{e^2}, e^{e^{e^2}}, 2^{\sqrt{2}}, 2^{\sqrt{3}} \]
are algebraically independent.

We shall use Schanuel's conjecture twice. We start with the numbers $1, \log 2,$ and $i\pi$ which are linearly independent over $\mathbb{Q}$ because $\log 2$ is irrational. Therefore, according to Schanuel's conjecture, three at least of the numbers
\[ 1, \log 2, i\pi, e, 2, -1 \]
are algebraically independent. This means that the three numbers $\log 2, \pi$ and $e$ are algebraically independent.

Therefore the 8 numbers
\[ 1, i\pi, \pi^2, e, e^2, \log 2, 2^{\frac{1}{3}} \log 2, 4^{\frac{1}{3}} \log 2 \]
are $\mathbb{Q}$-linearly independent. Again, Schanuel's conjecture implies that 8 at least of the numbers
\[ 1, i\pi, \pi^2, e, e^2, \log 2, 2^{\frac{1}{3}} \log 2, 4^{\frac{1}{3}} \log 2, e, -1, e^{\pi^2}, e, e^2, 2, 2^{\sqrt{2}}, 2^{\sqrt{3}} \]
are algebraically independent, and this means that the 8 numbers
\[ e, \pi, e^e, e^{e^2}, e^{e^{e^2}}, 2^{\sqrt{2}}, 2^{\sqrt{3}}, \log 2 \]
are algebraically independent.

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REFERENCES


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Hardy-Ramanujan Journal
You know that positive numbers are located on the right side of the number line, while the negative numbers are located on the left side. So let us now learn how to locate rational numbers on the number line. Let us study this in detail. Every one of you knows what a number line is. You know that positive numbers are located on the right side of the number line, while the negative numbers are located on the left side. So let us now learn how to locate rational numbers on the number line. Let us study this in detail. Suggested Videos. Introduction to Natural and Whole Numbers. Introduction to rational numbers. Properties of rational numbers. Rational Numbers on Number Line. The rational number are the numbers which can be represented on the number line. On Numbers and Games is a mathematics book by John Horton Conway first published in 1976. The book is written by a pre-eminent mathematician, and is directed at other mathematicians. The material is, however, developed in a playful and unpretentious manner and many chapters are accessible to non-mathematicians. Martin Gardner discussed the book at length, particularly Conway's construction of surreal numbers, in his Mathematical Games column in Scientific American in September 1976. Numbers on the Boards was the second single from My Name Is My Name. This intense, wordplay-dense track is a fan-favorite from MNIMN. Push says he wanted it to sound like a street record. He told me, "I'mma put it out in 15 minutes." I asked him, "Did you call the label?" Because we leak records all the time, we lie, we say we don't know how it happened, we got hacked, and it's always such a big issue. So I asked if he called them and Kanye said, "No, I didn't call them and I'm putting it out in 15 minutes." So, it dropped, man.