Variational Formulae for Codimension-One Foliations and Their Applications

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Abstract—The paper presents a new approach to study extrinsic geometry of codimension-one foliations. We start with formulae for the deformation of extrinsic geometric quantities as the Riemannian metric varies along the leaves. Then, we find variations of functionals depending on the principal curvatures of the leaves, in particular, for conformal along the leaves variations of metrics. Finally, we give applications to umbilical foliations, minimization of the total bending of the unit vector field, and to extrinsic geometric flows.

I. INTRODUCTION

The notion of a foliation appeared in the 1940’s in a series of papers of French mathematicians G. Reeb and Ch. Ehresmann, culminating in the book [5]. Since then, the Foliation theory has enjoyed a rapid development. In modern mathematics, a foliation is a geometric structure used to study manifolds, consisting of an integrable subbundle of the tangent bundle. Intuitively, a foliation corresponds to a decomposition of a manifold into a collection of connected submanifolds of the same dimension, called leaves, which locally look like pages of a book. By extrinsic geometry we mean properties of foliations on Riemannian manifolds which depend on the second fundamental form of the leaves and its invariants (principal curvatures, higher mean curvatures and so on).

In the paper we represent variational formulae which express variation of different quantities belonging to extrinsic geometry of a fixed foliation under leaf-wise variation of the Riemannian structure of the ambient manifold.

The problem of minimizing geometric quantities has been very popular for many years: recall, for example, classical isoperimetric inequalities, Fenchel estimates of total curvature of curves, works on tight and taut submanifolds. In the context of foliations, one has results of R. Langevin and co-authors ([2], [3] and so on), and [7]. In these cases, they consider a fixed Riemannian manifold and looks for geometric objects (submanifolds, foliations) minimizing geometric quantities defined usually as integrals of curvatures of different types.

On the other hand, there is some interest ([4], [9], and so on) in prescribing geometric quantities of given objects (say, foliations): given a foliated manifold \((M, F)\) and a geometric quantity \(Q\) (function, vector or tensor field) one may search for a Riemannian metric \(g\) on \(M\) for which a given geometric invariant (say, curvature of some sort) coincides with \(Q\).

The paper describes a new approach (of the author in collaboration with P. Walczak) combining the two just mentioned: given a foliated manifold \((M, F)\) and a geometric quantity \(Q\) (say, integral of a curvature-like invariant) we look for Riemannian metrics which minimize \(Q\) in the class of \(F\)-truncated metrics.

The developed variational formulae are applicable to extrinsic geometric flows on foliations, see [8]. These flows are defined as deformations of Riemannian metrics on a foliated manifold \(M\) subject to conditions expressed in terms of the second fundamental form of the leaves and its scalar invariants.

II. PRELIMINARIES

Let \(M^{n+1}\) be a closed (i.e., compact and without boundary) manifold with a Riemannian metric \(g\) and the Levi-Civita connection \(\nabla\). Let \(\mathcal{F}\) be a transversally oriented codimension-one foliation on \(M\), and \(N\) – the positively oriented unit normal of \(\mathcal{F}\). The notion of the \(F\)-truncated \((r, k)\)-tensor field \(\hat{S}\) \((r = 0, 1, \text{ and } \tau\) denotes the \(TF\)-component) will be helpful:

\[
\hat{S}(X_1, \ldots, X_k) = S(\tilde{X}_1, \ldots, \tilde{X}_k) \quad (X_i \in TM).
\]

Let \(\widehat{\mathfrak{L}}^k_1(M)\) be the bundle of \(\mathcal{F}\)-truncated \((k, 1)\)-tensors on \(M\). The inner product of tensors \(F, G \in \widehat{\mathfrak{L}}^k_1(M)\), denoted by \(\langle \cdot, \cdot \rangle\), is given by the following sum:

\[
\langle F, G \rangle = g^{a_1 b_1} \ldots g^{a_k b_k} g_{c_1 d_1} \ldots g_{c_k d_k} F^{c_1 \ldots c_k} g^{d_1 \ldots d_k}.
\]

The second fundamental form of (the leaves of) \(\mathcal{F}\) is a symmetric scalar \((0,2)\)-tensor field \(b\) given by

\[
b(X, Y) = g(\nabla_X Y, N), \quad X, Y \in TF.
\]

All the properties of \(\mathcal{F}\) which can be expressed in terms of \(b\) belong to extrinsic geometry. For example, a foliation \(\mathcal{F}\) is totally geodesic when \(b \equiv 0\); minimal when the mean curvature \(H = \frac{1}{n} \text{Tr}(b)\) vanishes; umbilical when \(b(X, Y) = H \cdot g(X, Y)\) for all \(X, Y \in TF\), and so on.

We denote by \(\mathcal{M} = \mathcal{M}(M, \mathcal{F}, N)\) the space of smooth Riemannian structures of finite volume on \(M\) with \(N\) being a unit normal to \(\mathcal{F}\). Elements of \(\mathcal{M}\) will be called \(\mathcal{F}\)-truncated metrics. Let \(\mathcal{M}_1 \subset \mathcal{M}\) be the subspace of metrics of unit...
volume. By $F\mathcal{M}_1$, we denote the foliation on $\mathcal{M}_1$ by leaves consisting of metrics conformally equivalent along $F$.

Let $A : X \in TF \mapsto -\nabla_X N$ be the Weingarten operator of the leaves, which we extend to a $(1,1)$-tensor field on $TM$ by $A(N) = 0$. Denote by $b_j$ the symmetric $(0,2)$-tensor fields on $M$ dual to powers $A^j$ of extended Weingarten operator,

$$\hat{b}_0 = \hat{g}, \quad \hat{b}_j(X,Y) = \hat{g}(A^j(X),Y), \quad j > 0.$$ 

*Power sums* of the principal curvatures $k_1, \ldots, k_n$ (the eigenvalues of $A$) are given by

$$\tau_j = k_1^j + \ldots + k_n^j = \text{Tr}(A^j), \quad j \geq 0.$$ 

These symmetric functions can be used using the elementary symmetric functions $\sigma_1, \ldots, \sigma_n$ (called *mean curvatures*).

$$\sigma_j = \sum_{i_1 < \ldots < i_j} k_{i_1} \cdots k_{i_j} \quad (0 \leq j \leq n).$$ 

Notice that $\sigma_0 = 1$, $\sigma_n = \det A$ and $\tau_n = n$. Evidently, the functions $\tau_{n+i}$ ($i > 0$), are not independent: they can be expressed as polynomials of $\vec{\tau} = (\tau_1, \ldots, \tau_n)$, using the Newton formulae.

Many authors investigated higher-order mean curvatures of hypersurfaces using the Newton transformations

$$T_r(A) = \sum_{i=0}^r (-1)^i \sigma_{r-i} A^i \quad (r < n) \quad (1)$$

of the shape operator. (By the Cayley-Hamilton Theorem, $T_n(A) = 0$.) Recently, Newton transformations have been applied successfully to foliations, see [6] and [7].

We study variational properties of the functional

$$I_f : g \in \mathcal{M} \mapsto \int_M f(\vec{\tau}) \text{dvol}_g, \quad f \in C^2(\mathbb{R}^n),$$

and of related functionals (total curvatures $\sigma_i$, $\tau_i$ etc.)

To shorten later formulas, we introduce the operator

$$\mathcal{V}(F) := \tau_1 F - \nabla_N F,$$

where $F \in \mathcal{A}^k(M)$. By the identity $\text{div} N = -\tau_1$ and the Divergence theorem, we have

$$\int_M \mathcal{V}(F) \text{dvol} = 0 \quad \text{for any} \quad F \in C^1(M). \quad (2)$$

The next lemmas are global and can be proved for arbitrary $(k,l)$-tensor fields. First, we show that the operator $\nabla_N$ is conjugate to $\mathcal{V}$.

**Lemma 1:** Let $S, B \in \mathcal{A}^2$. Then

$$\int_M \langle B, \nabla_N S \rangle \text{dvol} = \int_M \langle \mathcal{V}(B), S \rangle \text{dvol}. \quad (3)$$

In particular, for $F \in C^1(M)$ and $S = s \hat{g}$ we have

$$\int_M F N(s) \text{dvol} = \int_M s \mathcal{V}(F) \text{dvol}. \quad (4)$$

Given linear operator $\Phi : \mathcal{A}^2(M) \to \mathcal{A}^2(M)$, define

$$\mu(\Phi) = \inf_{S \in \mathcal{A}^2(M)} \int_M \langle \Phi(\nabla_N S), \nabla_N S \rangle \text{dvol} / \int_M \langle S, S \rangle \text{dvol}. \quad (5)$$

**Lemma 2:** Let $a$ be supremum of the lengths of $T.F^\perp$-geodesics. (i) If $a = \infty$ and $\langle \Phi(\nabla_N S), \nabla_N S \rangle \geq 0$ for any $S \in \mathcal{A}^2(M)$, then $\mu(\Phi) = 0$. (ii) If $0 \leq \langle \Phi(\nabla_N S), \nabla_N S \rangle \leq \|b\|^2 \langle S, S \rangle$ for any $S \in \mathcal{A}^2(M)$ and some $b \geq 0$, then $\mu(\Phi) \in [0, \pi^2 b^2 / a^2]$.

Concerning condition (i) of Lemma 2, notice that there exist compact Riemannian manifolds $(M^{n+1}, g), n > 2$, foliated by closed curves whose lengths are unbounded.

**Lemma 3:** Given linear operators $\Phi_i : \mathcal{A}^2(M) \to \mathcal{A}^2(M)$ ($i = 1, 2, 3$), define the functional

$$J(S) = \int_M \langle \Phi_1(S), S \rangle + \langle \Phi_2(\nabla_N S), \nabla_N S \rangle + \langle \Phi_3(S), \nabla_N S \rangle \text{dvol}.$$ 

If $J \geq 0$ for any symmetric tensor $S \in \mathcal{A}^2(M)$ then $\langle \Phi_2(\nabla_N S), \nabla_N S \rangle \geq 0$. Moreover, $J \geq 0$ when $\langle \Phi_2(\nabla_N S), \nabla_N S \rangle \geq 0$, and $\langle (\Phi_1 + \Phi_3) S, S \rangle \geq -\mu(\Phi_2) \langle S, S \rangle$.

**III. MAIN RESULTS**

**A. Variations of general functionals**

Consider a $t$-dependent metric $g_t$ such that $S = \partial_t g_t$ is an $F$-truncated $t$-dependent tensor. Since the difference of two connections is a tensor, $\Pi_t := \partial_t \nabla^t$ is a $(1,2)$-tensor field on $(M,g_t)$. Differentiating the classical formula for the Levi-Civita connection w. r. t. $t$ yields the known formula [10]

$$2g_t(\Pi_t(X,Y), Z) = \langle \nabla^t_X S(Y,Z), (\nabla^t_X Y)(Z) - (\nabla^t_Z Y)(X), Z \rangle,$$

where $X, Y, Z \in TM$. If $Y = Y(t)$ is time-dependent, then

$$\partial_t \nabla^t_X Y = \Pi_t(X,Y) + \nabla_X (\partial_t Y). \quad (6)$$

In order to calculate variations of the functional $I_f$ w. r. t. metrics $g_t \in \mathcal{M}_1$, we find the variational formula for $A$, and apply it to the Newton transformations $T_i(A)$ and to symmetric functions $\tau_j, \sigma_j$ of $A$.

**Proposition 1:** Let $g_t \in \mathcal{M}$ and $S = \partial_t g_t$. Then the Weingarten operator $A$ of $F$ and the functions $\tau_i$ and $\sigma_i$ of $A$ evolve by

$$\partial_t A = \frac{1}{2} \left( [A, S^t] - \nabla^t_N S^t \right),$$

$$\partial_t \tau_i = -\frac{i}{2} \text{Tr}(A^{i-1} \nabla^t_N S^t), \quad \partial_t \sigma_i = -\frac{1}{2} \text{Tr}(T_{i-1}(A) \nabla^t_N S^t).$$

For $F$-conformal variations $S = s \hat{g}$ $(s \in C^1(M))$ we have

$$\partial_t A = -\frac{1}{2} N(s) \hat{A}, \quad \partial_t \tau_i = -\frac{i}{2} \tau_{i+1} N(s), \quad \partial_t \sigma_i = -\frac{1}{2} (n - i + 1) \sigma_{i-1} N(s).$$

Next, we develop variational formulae for the functional $I_f$ restricted to metrics in $\mathcal{M}_1$. Let

$$\pi : \mathcal{M} \to \mathcal{M}_1, \quad \pi(g) = \hat{g} = (\text{vol}(M,g))^{-2/n} \hat{g} \oplus \hat{g}^\perp$$

be the $F$-conformal projection. Metrics $\tilde{g}_t = (\phi_t \hat{g}_t) \oplus g_t^\perp$ with dilating factors $\phi_t = \text{vol}(M,g_t)^{-2/n}$, belong to $\mathcal{M}_1$. 

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**References:** [6, 7, 10]
\[ I_1(g_t) = (1 + ct)^2 I_f(g), \quad I_1'(g) = \frac{n}{2} I_f(g). \]

We conclude that if \( g \) is a critical metric for \( I_f \), then functional \( g_t \in M_1 \) if and only if \( I_1(g_t) = 0 \).

In Theorem 1 below (and its corollaries), we find the variations of the functional \( I_f \) w.r.t. metrics \( g_t \in M_1 \).

Theorem 1: The gradient of the functional \( I_f : M_1 \to \mathbb{R} \), is given by

\[ \nabla I_f(g) = \frac{1}{2} \left( I_f(g) - \mathcal{F}(S_t) \right) \hat{\nabla} - \mathcal{V}(B_f), \]

where \( B_f = \sum_{i=1}^{N} \frac{\partial}{\partial f_i} \hat{\delta}_i \). The second variation of \( I_f(g_t) \) at a critical metric \( g = \tilde{g}_0 \) is given by

\[ I_f''(g) = \frac{1}{4} \left( f_I(g) - \mathcal{F}(S_t) \right) \hat{\nabla} - \mathcal{V}(B_f), \]

where \( \Phi_1(S_t) = \frac{1}{4} \left( f_I(g) - \mathcal{F}(S_t) \right) \hat{\nabla} - \mathcal{V}(B_f) \), \( \Phi_2(S_t) = \frac{1}{4} \left( f_{i,j} - \mathcal{F}(S_t) \right) \hat{\nabla} - \mathcal{V}(B_f) \), \( \Phi_3(S_t) = \frac{1}{4} \left( f_{i,j} - \mathcal{F}(S_t) \right) \hat{\nabla} - \mathcal{V}(B_f) \), \( \Phi_4(S_t) = \frac{1}{4} \left( f_{i,j} - \mathcal{F}(S_t) \right) \hat{\nabla} - \mathcal{V}(B_f) \).

For \( \nabla \)-conformal variations, \( S = s \hat{g}, s : M \to \mathbb{R} \), we have

\[ I_f'(g_t)_{t=0} = \frac{1}{4} \int_M \Phi_f N(s)^2 d\nu, \]

where \( \Phi_f = \sum_{i=1}^{N} \frac{\partial}{\partial f_i} \hat{\delta}_i \).

Example 1 (Totally geodesic foliations): Let \( \mathcal{F} \) be a totally geodesic foliation on \( (M, g) \) of unit volume. Then

\[ A = 0, \quad \mathcal{F} = 0, \quad B_f = \frac{1}{2} f_{i,1}(0) \hat{\nabla} - \mathcal{V}(B_f), \quad I_f'(g) = f(0). \]

By (8), \( g \) is a critical metric for \( I_f \). We have \( \Phi_f = \frac{n}{2} f_{i,1}(0) + \frac{n}{4} f_{i,1}(0) \)

\[ \Phi_1(S_t) = 0, \quad \Phi_3(S_t) = \frac{1}{2} f_{i,1}(0)(\nabla S_t^2) \hat{\nabla}, \]

\[ \Phi_2(S_t) = \frac{1}{4} f_{i,1}(0)(\nabla S_t^2) \hat{\nabla} + \frac{1}{2} f_{i,1}(0) S_t, \]

Using (2) and Lemma 1, we calculate

\[ I_f''(g) = \int_M \left( \frac{f_{i,1}(0)}{4} N(\nabla S_t^2)^2 + \frac{f_{i,1}(0)}{2} Tr(\nabla S_t^2)^2 \right) d\nu. \]

We conclude that \( I_f'' \geq 0 \), and \( f_{i,1}(0) \geq 0 \) and \( f_{i,2}(0) \geq 0 \).

B. Variations of total mean curvatures

Consider functionals on \( M_1 \) for particular cases of \( f \):

\[ I_{r,k}(g) = \int_M \tau_k d\nu, \quad I_{r,s,k}(g) = \int_M \sigma_k d\nu, \quad k > 0. \]

It is known [5] that \( I_{r,1} = I_{r,1} = 0 \) for any \( \mathcal{F} \) and \( g \).

From Theorem 1 it follows

Corollary 1: The gradients of functionals (10) are given by

\[ \nabla I_{r,k}(g) = 2 \left( \tau_k - \mathcal{F}(S_t) \right) \hat{\nabla} - \mathcal{V}(B_f), \]

\[ \nabla I_{r,s,k}(g) = 2 \left( \sigma_k - \mathcal{F}(S_t) \right) \hat{\nabla} - \mathcal{V}(B_f). \]

The second variations of \( I_{r,k} \) and \( I_{r,s,k} \) at a critical metric \( g = \tilde{g}_0 \) are given by (9), where for \( f = \tau_k \)

\[ \Phi_1(S_t) = \frac{k}{4} \left( \tau_k - \mathcal{F}(S_t) \right) \hat{\nabla} - \frac{1}{2} Tr(S_t^2) A\}, \]

\[ \Phi_2(S_t) = \frac{k}{4} \left( \tau_k - \mathcal{F}(S_t) \right) \hat{\nabla} - \frac{1}{2} Tr(S_t^2) A\}

and for \( f = \sigma_k \) we have

\[ \Phi_1(S_t) = \frac{k}{4} \left( \sigma_k - \mathcal{F}(S_t) \right) \hat{\nabla} - \frac{1}{2} Tr(S_t^2) A\}

\[ \Phi_2(S_t) = \frac{k}{4} \left( \sigma_k - \mathcal{F}(S_t) \right) \hat{\nabla} - \frac{1}{2} Tr(S_t^2) A\}

For \( \mathcal{F} \)-conformal variations, \( S = s \hat{g}, s \in M \to \mathbb{R} \), we have, resp.

\[ \Phi_f = k(\bar{k} - 1) \tau_k - 1, \quad \Phi_f = (n-k+1)(n-k+2) \sigma_k - 1. \]

C. Variational formulae for umbilical foliations

Let \( \mathcal{F} \) be a umbilical foliation on \( (M, g) \) with the normal curvature \( \lambda : M \to \mathbb{R} \) (i.e., \( \lambda = H = \frac{1}{n} \tau_1 \)). One may show that \( \mathcal{F} \)-conformal variations \( g_t \in M \) preserve this property.

Lemma 4: Let \( \mathcal{F} \) be a umbilical foliation on \( (M, g_0) \). If \( g_t \in M \) \( (0 < t < 1) \) is an \( \mathcal{F} \)-conformal variation of \( g_0 \), then \( \mathcal{F} \) is umbilical for any \( g_t \).

Given function \( \psi \in C^2(\mathbb{R}) \), consider the functional \( I_\psi : M \to \mathbb{R} \) over the space \( U \) of all Riemannian metrics w.r.t. which \( \mathcal{F} \) is umbilical.

\[ I_\psi(g) = \int_M \psi(\lambda) d\nu, \]

From the above and Theorem 1 we obtain the following

Corollary 2: Let \( \mathcal{F} \) be a umbilical foliation on \( (M, g) \), and \( \psi \in C^2(\mathbb{R}) \). The \( \mathcal{F} \)-conformal component of the gradient of the functional \( I_\psi \) is given by

\[ \nabla \mathcal{F} I_\psi(g) = \left( \frac{1}{2} \psi(\lambda) - I_\psi(g) - \frac{1}{n} \psi'(\lambda) \right) \hat{\nabla}. \]
The second variation of $I_\psi$ at a critical metric $g = \tilde{g}_0 \in \mathcal{M}_1$, w. r. t. $\mathcal{F}$-conformal variations $\tilde{g}_t \in \mathcal{M}_1$ with $S = s \tilde{g}_t$ ($s \in C^1(M)$) is

$$I_\psi'(\tilde{g}_t)_{|t=0} = \frac{1}{4} \frac{\partial}{\partial s} \int_M \psi''(s) N(s)^2 \text{dvol}. \quad (13)$$

From Theorem 1 and Lemmas 2, 3 we deduce the following

Theorem 2: Let a metric $g$ be a stable local maximum for $I_\psi$ (with a fixed $f \in C^2(\mathbb{R}^n)$) on the space $\mathcal{M}_1$. If

$$\int_M \langle \Phi_g(V(S)), \nabla N(S) \rangle \text{dvol} \geq 0$$

for any $\mathcal{F}$-truncated symmetric $(0, 2)$-tensor $S$, then $\mathcal{F}$ is umbilical.

D. The bending of the unit normal vector field

The energy of a unit vector field $N$ on $(M^{n+1}, g)$ can be expressed by the formula

$$E_N(g) = \frac{1}{2} (n+1) \text{vol}(M, g) + \int_M ||\nabla N||^2 \text{dvol},$$

see, for example, [1]. The last integral,

$$B_N(g) = \int_M ||\nabla N||^2 \text{dvol},$$

(up to the constant) is called the total bending of $N$. The problems of minimizing $E_N(g)$ and $B_N(g)$ w. r. t. variations $\tilde{g}_t \in \mathcal{M}_1$ are equivalent. One may decompose the bending:

$$B_N = I_{\tau, 2} + B_N^1,$$

where $B_N^1(g) = \int_M ||Z||^2 \text{dvol}$.

Lemma 5: The vector field $Z = \nabla N$ is evolved by $g_t \in \mathcal{M}$ with $S = \partial_t g_t$ as

$$\partial_t Z = -S^t(Z). \quad (15)$$

Hence, variations $g_t \in \mathcal{M}$ preserve Riemannian foliations.

Since the components of 1-form $Z^p = g(Z, \cdot)$ dual to $Z$ are $(Z^p)_i = Z^p g_{ia}$, by definition of the tensor product, we have

$$(Z^p \odot Z^q)_{ij} = Z^p g_{ia} Z^q g_{jb}.$$

Theorem 3: The gradient of the bending functional $B_N : \mathcal{M}_1 \to \mathbb{R}$ is given by

$$\nabla B_N(g) = \frac{1}{2} (||Z||^2_{g} + \tau_2 - B_N(g)) \tilde{g} - Z^p \odot Z^q - \nabla(\tilde{b}_1),$$

where $Z = \nabla N$. The second variation of $B_N$ at a critical metric $g = \tilde{g}_0$, where $S = \partial_t \tilde{g}_t$, is given by (9), where

$$\Phi_1(S^t) = -\frac{1}{4} (||Z||^2_{g} + \tau_2 - B_N(g)) (\text{Tr} S^t) \tilde{i} \tilde{d},$$

$$\Phi_2(S^t) = \frac{1}{2} S^t, \quad \Phi_3(S^t) = \text{Tr} (A S^t) \tilde{i} \tilde{d},$$

For $\mathcal{F}$-conformal variations, $S = s \tilde{g}_t$, $s : M \to \mathbb{R}$, we have

$$B_N^1(\tilde{g}_t)_{|t=0} = \frac{n}{2} \int_M \left( \frac{n}{2} - 1 \right) ||Z||^2_{g} s^2 + N(s)^2 \text{dvol}. \quad (14)$$

E. Extrinsic Geometric Flows

Given $f_j \in C^2(\mathbb{R}^n)$, let $h(b)$ be the $(0, 2)$-tensor dual to

$$h(A) := \sum_{j=0}^{n-1} f_j(\tilde{T}) A^j.$$

In [8] we found sufficient conditions for existing and uniqueness of solutions $g_t$ for $t \in (0, \infty)$ to the PDE

$$\partial_t g_t = h(b_t) - \frac{\rho_t}{\tau_1} g_t$$

called the normalized Extrinsic Geometric Flow. The choice of $h(b)$ is natural; the powers $b_j$ are the only $(0, 2)$-tensors which can be obtained algebraically from $b$, while $\tau_1, \ldots, \tau_n$ generate all scalar invariants of extrinsic geometry.

By Theorem 1 with $f = \text{Tr} h(A)$ and Corollary 2, we have

Theorem 4: (a) Let $g_t \in \mathcal{M}_1 (t \geq 0)$ be the solution of (16) with $h(b) = \tau_1^k b_1$ for some $k \in \mathbb{N}$. Then $g_t$ approach (in a weak sense) a metric making $\mathcal{F}$ a minimal foliation: the functional $\int_M \tau_1^{k+1} \text{dvol} \rightarrow 0$ as $t \rightarrow \infty$.

(b) If $g_t \in \mathcal{M}_1 (t \geq 0)$ is a solution to the PDE

$$\partial_t g_t = (\lambda^{-b} - \rho_t) g_t,$$

and $\mathcal{F}$ is umbilical w. r. t. $g_t$, then $\mathcal{F}$ is umbilical w. r. t. $g_t$ and $g_t$ approach (in a weak sense) a metric making $\mathcal{F}$ totally geodesic: the functional $\int_M \lambda_t^2 \text{dvol} \rightarrow 0$ as $t \rightarrow \infty$.

IV. Conclusion

The following results for $\mathcal{F}$-truncated variations of metrics on foliations are obtained: (i) formulae for the deformation of extrinsic geometric quantities of foliations; (ii) variation formulae for the functionals depending on the principal curvatures of the leaves. They are applied to umbilical foliations, minimizing the total energy and bending, and also to extrinsic geometric flows, which may provide more results on geometry of foliations.

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REFERENCES

Integral formulae for foliated Riemannian manifolds provide obstructions for existence of foliations or compact leaves of them with given geometric properties. This paper continues our recent study and presents new integral formulae and their applications for codimension-one foliated Randers spaces. The goal is a generalization of Reeb's formula (that the total mean curvature of the leaves is zero) and its companion (that twice total second mean curvature of the leaves equals to the total Ricci curvature in the normal direction). We also extend results by Brito, Langevin and Rosenberg (that total mean curvatures of arbitrary order for a codimension-one foliated Riemannian manifold of constant curvature don't depend on a foliation). Integral formulae for codimension-one foliated Finsler manifolds. One of such formulae was used in [7] to prove that codimension-one foliations of a closed Riemannian manifold of negative Ricci curvature are far (in a sense dened there) from being umbilical. In this paper we study extrinsic geometry of a codimension-one transversely orien- ted foliation $F$ of a closed Finsler space $(M, F)$, in particular, of a Randers space $(M, \|\cdot\| + \|\cdot\|_F)$, $\|\cdot\|$ being the norm of a Riemannian structure $a$ and $\|\cdot\|_F$ a 1-form of $\|\cdot\|$-norm smaller than 1 everywhere on $M$. Using a unit normal. In this section we apply the variational approach to nd a relationship between the Riemann curvature of $F$ and $g$. It generalizes the following. actions on the circle and real codimension one foliations. 11:35 â€“ 12:05 j a. a. L. A, A trace formula for foliated ows on foliations. of codimension one. 12:15 â€“ 12:45 M. C. umbilical foliations in hyperbolic spaces. Integral formulae for higher order mean. curvatures of foliations. 16:00 â€“ 16:30 C. umbilical foliations in hyperbolic spaces. Uniform perfectness of dieomorphism groups and its application. 12:45 â€“ 14:15 L. 14:30 â€“ 18:00 s., Rho-invariants for foliations and their stability proper-. 16:00 â€“ 16:30 C. 16:30 â€“ 17:00 h.